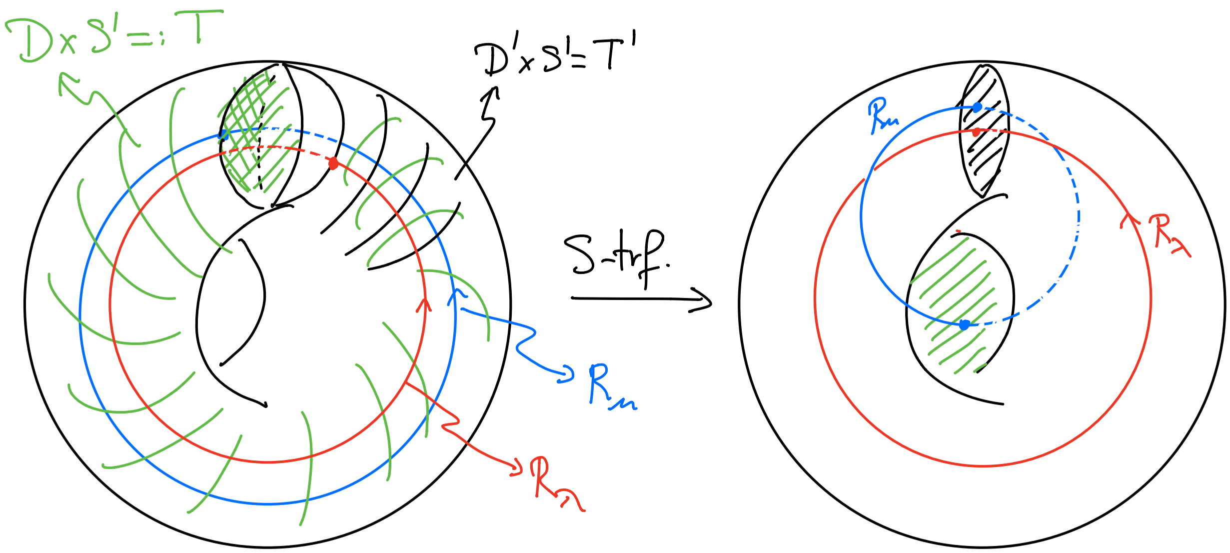
Proof of Verlinde's formula:

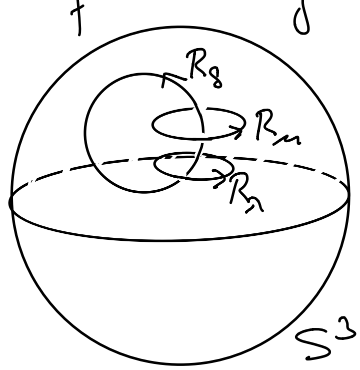
Consider the situation of performing surgery on $S^2 \times S^1$ with two Wilson lines:



$S^2 \times S^1$ $\qquad\qquad$ S^3

$$Z(S^3; L(R_\mu, R_\lambda)) = \sum_\nu S_\mu{}^\nu \underbrace{Z(S^2 \times S^1; R_\nu, R_\lambda)}_{=\,\delta_{\nu\lambda}} = S_{\mu\lambda}$$

Letting 3 Wilson lines pass through S^2, we get after surgery S^3 with link



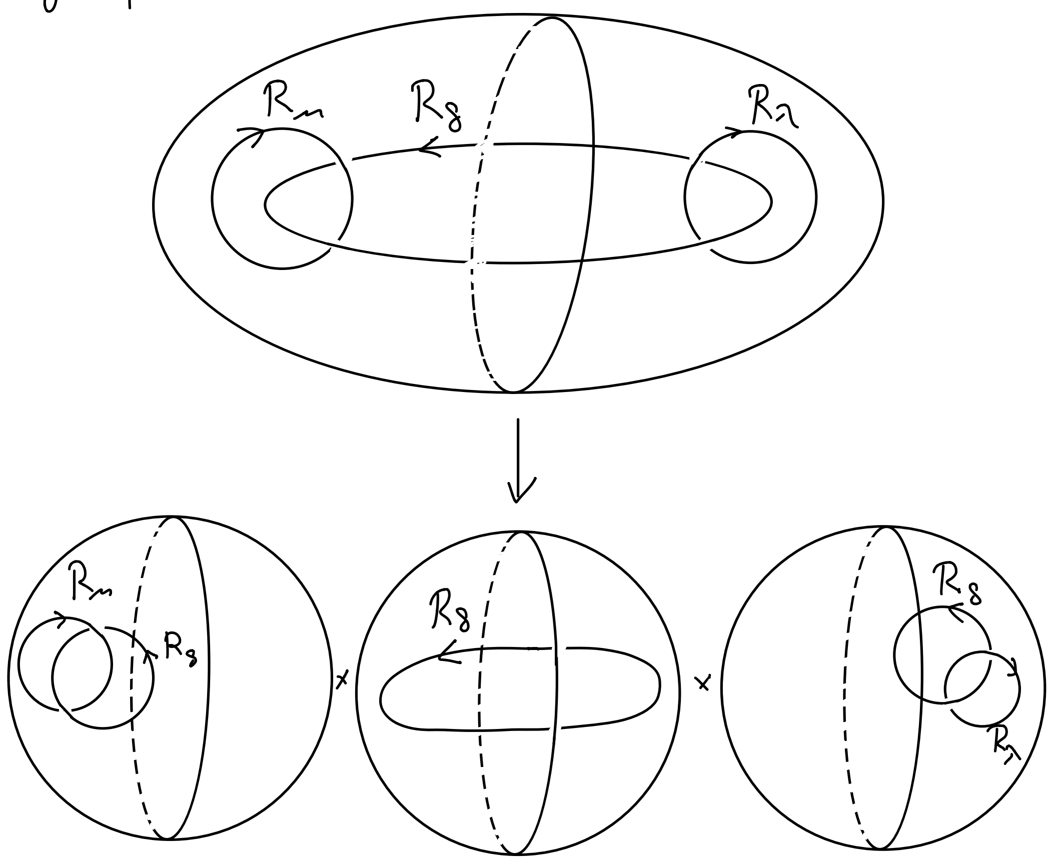
$$\longrightarrow Z(S^3; L(R_\delta, R_\mu, R_\lambda))$$

This time we get

$$Z(S^3; L(R_\gamma, R_\mu, R_\lambda)) = \sum_\nu S_\gamma{}^\nu Z(S^2 \times S^1; R_\nu, R_\mu, R_\lambda)$$

$$= \sum_\nu S_\gamma{}^\nu N_{\nu\mu\lambda} \quad (*)$$

Using factorization,



we have

$$Z(S^3, L(R_\gamma, R_\mu, R_\lambda)) \cdot Z(S^3; R_\gamma)$$

$$= Z(S^3; L(R_\gamma, R_\mu)) \cdot Z(S^3; L(R_\gamma, R_\lambda))$$

$$\longrightarrow Z(S^3; L(R_\gamma, R_\mu, R_\lambda)) = S_{\gamma\mu} S_{\gamma\lambda} / S_{0,\gamma}$$

Combining with (*), we finally get

$$\frac{S_{\delta\mu} S_{\delta\lambda}}{S_{0\delta}} = \sum_{\nu} S_{\delta}{}^{\nu} N_{\nu\mu\lambda}$$

Projective representations of mapping class groups

Let us summarize in a systematic way what we have learned.

A topological quantum field theory (TQFT) in d+1 dimensions is a functor Z satisfying the following conditions:

1. Σ d-dim mfd., $\partial\Sigma = \phi$
   $\longrightarrow$ finite-dim. compl. vector space $\mathcal{H}_{\Sigma}$

2. Y (d+1)-dim mfd., $\partial Y = \Sigma$
   $\longrightarrow$ a vector $Z(Y) \in \mathcal{H}_{\Sigma}$

Z has to satisfy the following properties:

A1) $-\Sigma$ orientation reversed Σ
   $\rightarrow \mathcal{H}_{-\Sigma} = \mathcal{H}_{\Sigma}^{*}$ (dual of $\mathcal{H}_{\Sigma}$)

A2) For a disjoint union $\Sigma_1 \cup \Sigma_2$ we have

$$\mathcal{H}_{\Sigma_1 \cup \Sigma_2} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}$$

$A_1) + A_2)$ imply:

$(d+1)$-mfd Y with $\partial Y = (-\Sigma_1) \cup \Sigma_2$

$$\longrightarrow Z(Y) \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_{\Sigma_1}, \mathcal{H}_{\Sigma_2})$$

"cobordism" between Σ_1 and Σ_2

A3) Composition of cobordisms $\partial Y_1 = (-\Sigma_1) \cup \Sigma_2$ and $\partial Y_2 = (-\Sigma_2) \cup \Sigma_3$

$$\longrightarrow Z(Y_1 \cup Y_2) = Z(Y_2) \circ Z(Y_1)$$

A4) For an empty set $\emptyset$ we have $Z(\emptyset) = \mathbb{C}$.

A5) For the closed unit interval I, $Z(\Sigma \times I)$ is the identity map on $\mathcal{H}_\Sigma$.

Let $\mathcal{C}$ denote the category with

- objects: compact oriented smooth mfds. without boundary
- morphisms: cobordisms

$\rightarrow$ Z is a functor between $\mathcal{C}$ and the linear category with objects finite-dim. complex vector spaces and morphisms linear maps.

Define: $\mathrm{Diff}^+(\Sigma) \cong$ orientation preserving diffeomorphisms of Σ.

$\xrightarrow{A3-A5}$ $\mathrm{Diff}^+(\Sigma)$ acts linearly on $\mathcal{H}_\Sigma$.

$\longrightarrow$ if $\partial Y = \phi \longrightarrow Z(Y) \in \mathbb{C}$

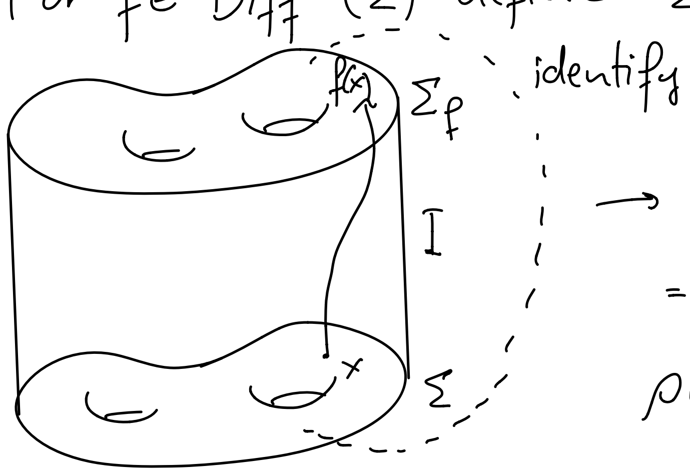
in particular: $Z(Y) = \langle Z(Y_+), Z(Y_-) \rangle$

where $Y = Y_+ \cup Y_-$, $\partial Y_+ = \Sigma = -\partial Y_-$

$Z(Y_+) \in \mathcal{H}_\Sigma$, $Z(Y_-) \in \mathcal{H}_\Sigma^*$

By A3 $Z(Y)$ is independent of a decomposition of Y.

For $f \in \mathrm{Diff}^+(\Sigma)$ define Σ_f



$\longrightarrow Z(\Sigma_f)$

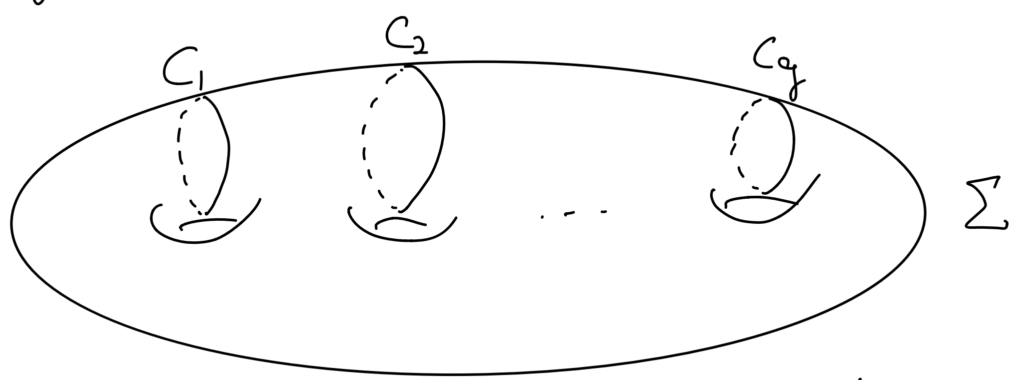
$= \mathrm{Tr}\, \rho(f)$ where

$\rho(f): \mathcal{H}_\Sigma \longrightarrow \mathcal{H}_{\Sigma_f}$

For the case where $f = id$:

$$\dim \mathcal{H}_{\Sigma} = Z(\Sigma \times S^1)$$

Now let Σ be a closed oriented surface of genus g:



Introduce equivalence rel. on $\mathrm{Diff}^+(\Sigma)$:

$h \sim h'$: h and $h' \in \mathrm{Diff}^+(\Sigma)$ are isotopic

$\longrightarrow$ $\mathrm{Diff}^+(\Sigma)/\sim$ forms a group "mapping class group" $\mathcal{M}_g$.

In the case of $g=0$ and n marked points $\longrightarrow$ $\mathcal{M}_{0,n}$ gives the braid group

Want to generalize to higher g.

Cut out Σ by $C_1, \dots, C_g$ $\to$ 2-sphere with $2g$ boundary comp.

→ consider space of conformal blocks of Riemann sphere with $2g$ marked points $p_1, p_2, \ldots, p_{2g-1}, p_{2g}$

consider $\mathfrak{g} = sl_2(\mathbb{C})$ with fixed $K \in \mathbb{Z}_+$.

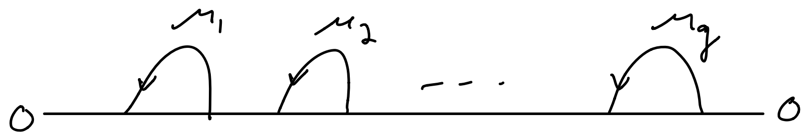
→ associate level K highest weights $\mu_1, \mu_1^*, \ldots, \mu_g, \mu_g^*$ (λ is h.w. rep. of H_λ)

(and H_0 to the origin, H_0^* to ∞)

$$\hookrightarrow \mathrm{Hom}_{\mathfrak{g}}\left(V_{\mu_1} \otimes V_{\mu_1}^* \otimes \cdots \otimes V_{\mu_g} \otimes V_{\mu_g}^*, \mathbb{C}\right)$$
$$\shortparallel$$
$$V_{\mu_1 \mu_1^* \cdots \mu_g \mu_g^*}$$

set
$$V_\Sigma = \bigoplus_{\mu_1, \mu_1^*, \ldots, \mu_g, \mu_g^* \in P_+(K)} V_{\mu_1 \mu_1^* \cdots \mu_g \mu_g^*}$$

→ admissible labelings for



Lemma 3:

$$\dim V_\Sigma = \sum_{\lambda \in P_+(K)} \left(\frac{1}{S_{0\lambda}}\right)^{2g-2}$$

Proof: Follows from Prop. 7. §6. □